Weak and strong k-connectivity games

Asaf Ferber * Dan Hefetz †

February 27, 2013

Abstract

For a positive integer k we consider the k-vertex-connectivity game, played on the edge set of K_n, the complete graph on n vertices. We first study the Maker-Breaker version of this game and prove that, for any integer $k \geq 2$ and sufficiently large n, Maker has a strategy for winning this game within $\lfloor kn/2 \rfloor + 1$ moves, which is clearly best possible. This answers a question from [5]. We then consider the strong k-vertex-connectivity game. For every positive integer k and sufficiently large n, we describe an explicit first player's winning strategy for this game.

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the *strong game* $(X, \mathcal{F})$, two players, called Red and Blue, take turns in claiming one previously unclaimed element of X, with Red going first. The winner of the game is the first player to fully claim some $F \in \mathcal{F}$. If neither player is able to fully claim some $F \in \mathcal{F}$ by the time every element of X has been claimed by some player, the game ends in a *draw*. The set X will be referred to as the *board* of the game and the elements of $\mathcal{F}$ will be referred to as the *winning sets*.

It is well known from classic Game Theory that, for every strong game $(X, \mathcal{F})$, either Red has a winning strategy (that is, he is able to win the game against any strategy of Blue) or Blue has a drawing strategy (that is, he is able to avoid losing the game against any strategy of Red; a *strategy stealing* argument shows that Blue cannot win the game). For certain games, a hypergraph coloring argument can be used to prove that draw is impossible and thus these games are won by Red. However, the aforementioned arguments are purely existential. That is, even if it is known that Red has a winning strategy for some strong game $(X, \mathcal{F})$, it might be very hard to describe such a strategy explicitly. The few examples of natural games for which an explicit winning strategy is known include the *perfect matching* and *Hamilton cycle* games (see [2]).

Partly due to the great difficulty of studying strong games, weak games were introduced. In the *Maker-Breaker game* (also known as *weak* game) $(X, \mathcal{F})$, two players, called Maker and

*School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: ferberas@post.tau.ac.il

†School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom. Email: d.hefetz@bham.ac.uk

Breaker, take turns in claiming previously unclaimed elements of X, with Breaker going first (in some cases it will be convenient to assume that Maker starts the game; whenever this assumption is made, it will be stated explicitly). Each player claims **exactly** one element of X per turn (sometimes it will be convenient to assume that each player claims **at most** one element of X per turn; since Maker-Breaker games are bias monotone, this has no effect on the outcome of the game). The set X is called the *board* of the game and the members of $\mathcal{F}$ are referred to as the *winning sets*. Maker wins the game as soon as he occupies all elements of some winning set. If Maker does not fully occupy any winning set by the time every board element is claimed by some player, then Breaker wins the game. Note that being the first player is never a disadvantage in a Maker-Breaker game. Hence, in order to prove that Maker can win some Maker-Breaker game as the first or second player, it suffices to prove that he can win this game as the second player.

In this paper we study the weak and strong versions of the k-vertex-connectivity game $(E(K_n), \mathcal{C}_n^k)$. The board of this game is the edge set of the complete graph on n vertices and its family of winning sets $\mathcal{C}_n^k$, consists of the edge sets of all k-vertex-connected subgraphs of K_n.

It is easy to see (and also follows from [7]) that, for every $n \geq 4$, Maker can win the weak game $(E(K_n), \mathcal{C}_n^1)$ within $n-1$ moves. Clearly this is best possible. It follows from [6] that, if n is not too small, then Maker can win the weak game $(E(K_n), \mathcal{C}_n^2)$ within $n+1$ moves and this is best possible as well. It was proved in [5] that, for every fixed $k \geq 3$ and sufficiently large n, Maker can win the weak game $(E(K_n), \mathcal{C}_n^k)$ within $kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n)$ moves. Since, clearly Maker cannot win this game in less than $kn/2$ moves, this shows that the number of excess moves Maker plays is $o(n)$. It was asked in [5] whether the dependency in n of the number of excess moves can be omitted, that is, whether Maker can win $(E(K_n), \mathcal{C}_n^k)$ within $kn/2 + c_k$ moves for some c_k which is independent of n. We answer this question in the affirmative.

Theorem 1.1 *let $k \geq 2$ be an integer and let n be a sufficiently large integer. Then Maker (as the first or second player) has a strategy for winning the weak game $(E(K_n), \mathcal{C}_n^k)$ within at most $\lfloor kn/2 \rfloor + 1$ moves.*

The upper bound on the number of moves obtained in Theorem 1.1 is clearly best possible.

In the minimum-degree-k game $(E(K_n), \mathcal{D}_n^k)$, the board is again the edge set of K_n and the family of winning sets $\mathcal{D}_n^k$, consists of the edge sets of all subgraphs of K_n with minimum degree at least k. Since $\mathcal{C}_n^k \subseteq \mathcal{D}_n^k$ for every k and n we immediately obtain the following result.

Corollary 1.2 *Let $k \geq 1$ be an integer and let n be a sufficiently large integer. Then Maker (as the first or second player) has a strategy to win the weak game $(E(K_n), \mathcal{D}_n^k)$ within at most $\lfloor kn/2 \rfloor + 1$ moves.*

It is easy to see that Maker cannot win $(E(K_n), \mathcal{D}_n^k)$ within $\lfloor kn/2 \rfloor$ moves. Hence, the bound stated in Corollary 1.2 is tight.

Note that, for $k = 1$, Corollary 1.2 does not follow from Theorem 1.1. However, this case was proved in [5]. Moreover, we will prove a strengthening of this result in Section 3.

It was observed in [2] that a fast winning strategy for Maker in the weak game $(X, \mathcal{F})$ has the potential of being used to devise a winning strategy for the first player in the strong game

$(X, \mathcal{F})$. Using our strategy for the weak game $(E(K_n), \mathcal{C}_n^k)$, we will devise an explicit winning strategy for the corresponding strong game. We restrict our attention to the case $k \geq 3$ as the (much simpler) cases $k = 1$ and $k = 2$ were discussed in [2].

Theorem 1.3 *let $k \geq 3$ be an integer and let n be a sufficiently large integer. Then Red has a strategy to win the strong game $(E(K_n), \mathcal{C}_n^k)$ within at most $\lfloor kn/2 \rfloor + 1$ moves.*

Our proof of Theorem 1.3 will in fact show that Red can build a k-vertex-connected graph before Blue can build a graph with minimum degree at least k. We thus have the following corollary.

Corollary 1.4 *Let $k \geq 1$ be an integer and let n be a sufficiently large integer. Then Red has a strategy to win the strong game $(E(K_n), \mathcal{D}_n^k)$ within at most $\lfloor kn/2 \rfloor + 1$ moves.*

As with Corollary 1.2, the cases $k = 1$ and $k = 2$ do not follow from Theorem 1.3. However, these simple cases were discussed in [2]. Moreover, for $k = 1$ we will prove a strengthening of this result in Section 3.

The rest of this paper is organized as follows: in Subsection 1.1 we introduce some notation and terminology that will be used throughout this paper. In Section 2 we describe a family of k-vertex-connected graphs that will be used in the proofs of Theorems 1.1 and 1.3. In Section 3 we study certain simple games; the results obtained will be used in the following sections. In Section 4 we prove Theorem 1.1 and in Section 5 we prove Theorem 1.3. Finally, in Section 6 we present some open problems.

1.1 Notation and terminology

Our graph-theoretic notation is standard and follows that of [8]. In particular, we use the following.

For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and one endpoint in B, and let $e_G(A, B) = |E_G(A, B)|$. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G which is induced on the set S. For disjoint sets $S, T \subseteq V(G)$, let $N_G(S, T) = \{u \in T : \exists v \in S, uv \in E(G)\}$ denote the set of neighbors of the vertices of S in T. For a set $T \subseteq V(G)$ and a vertex $w \in V(G) \setminus T$ we abbreviate $N_G(\{w\}, T)$ to $N_G(w, T)$, and let $d_G(w, T) = |N_G(w, T)|$ denote the degree of w into T. For a set $S \subseteq V(G)$ and a vertex $w \in V(G)$ we abbreviate $N_G(S, V(G) \setminus S)$ to $N_G(S)$ and $N_G(w, V(G) \setminus \{w\})$ to $N_G(w)$. We let $d_G(w) = |N_G(w)|$ denote the degree of w in G. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. For vertices $u, v \in V(G)$ let $dist_G(u, v)$ denote the *distance* between u and v in G, that is, the number of edges in a shortest path of G, connecting u and v. Often, when there is no risk of confusion, we omit the subscript G from the notation above. For a positive integer k, let $[k]$ denote the set $\{1, \ldots, k\}$.

Assume that some Maker-Breaker game, played on the edge set of some graph G, is in progress. At any given moment during this game, we denote the graph spanned by Maker's edges by M

and the graph spanned by Breaker's edges by B. At any point during the game, the edges of $G \setminus (M \cup B)$ are called *free*.

Similarly, assume that some strong game, played on the edge set of some graph G, is in progress. At any given moment during this game, we denote the graph spanned by Red's edges by R and the graph spanned by Blue's edges by B. At any point during the game, the edges of $G \setminus (R \cup B)$ are called *free*.

2 A family of k-vertex-connected graphs

In this section we describe a family of k-vertex-connected graphs. We will use this family in the proofs of Theorem 1.1 and Theorem 1.3.

Let $k \geq 3$ be an integer and let n be a sufficiently large integer. Let $\mathcal{G}_k$ be the family of all graphs $G_k = (V, E_k)$ on n vertices for which there exists a partition $V = V_1 \cup \ldots \cup V_{k-1}$ such that all of the following properties hold:

(i) $|V_i| \geq 5$ for every $1 \leq i \leq k-1$.

(ii) $\delta(G_k) \geq k$.

(iii) $G_k[V_i]$ admits a Hamilton cycle C_i for every $1 \leq i \leq k-1$.

(iv) For every $1 \leq i < j \leq k-1$ the bipartite subgraph of G_k with parts V_i and V_j admits a matching of size 3.

(v) For every $1 \leq i \leq k-1$ and every $u \in V_i$, $|\{j \in [k-1] \setminus \{i\} : d_{G_k}(u, V_j) = 0\}| \leq 1$.

(vi) For every $1 \leq i \leq k-1$ and every $u, v \in V_i$, if $|\{j \in [k-1] \setminus \{i\} : d_{G_k}(u, V_j) = 0\}| = |\{j \in [k-1] \setminus \{i\} : d_{G_k}(v, V_j) = 0\}| = 1$, then $dist_{C_i}(u, v) \geq 2$.

Proposition 2.1 *For every integer $k \geq 3$ and sufficiently large integer n, every $G_k \in \mathcal{G}_k$ is k-vertex-connected.*

Proof Let G_k be any graph in $\mathcal{G}_k$. Let $S \subseteq V$ be an arbitrary set of size $k-1$. We will prove that $G_k \setminus S$ is connected. We distinguish between the following three cases.

Case 1: $|S \cap V_i| = 1$ for every $1 \leq i \leq k-1$.
Since $G_k[V_i]$ is Hamiltonian for every $1 \leq i \leq k-1$ by Property (iii) above, it follows that $(G_k \setminus S)[V_i]$ is connected for every $1 \leq i \leq k-1$. Hence, in order to prove that $G_k \setminus S$ is connected, it suffices to prove that $E_{G_k \setminus S}(V_i, V_j) \neq \emptyset$ holds for every $1 \leq i < j \leq k-1$. Fix some $1 \leq i < j \leq k-1$. It follows by Property (iv) above that there exist vertices $x_i, y_i, z_i \in V_i$ and $x_j, y_j, z_j \in V_j$ such that $x_i x_j, y_i y_j, z_i z_j \in E_{G_k}(V_i, V_j)$. Clearly, at least one of these edges is present in $G_k \setminus S$.

Case 2: There exist $1 \leq i < j \leq k-1$ such that $S \cap V_i = \emptyset$ and $S \cap V_j = \emptyset$.
It follows by Properties (iii) and (iv) above that $(G_k \setminus S)[V_i \cup V_j]$ is connected. Moreover, it follows by Property (v) above that $V_i \cup V_j$ is a dominating set of G_k. Hence, $G_k \setminus S$ is connected in this case.

Case 3: There exist $1 \leq i \neq j \leq k-1$ such that $S \cap V_i = \emptyset$, $|S \cap V_j| = 2$ and $|S \cap V_t| = 1$ for every $t \in [k-1] \setminus \{i,j\}$.

It follows by Property (iii) above that $(G_k \setminus S)[V_i]$ is connected. Hence, in order to prove that $G_k \setminus S$ is connected, it suffices to prove that, for every vertex $u \in V \setminus (V_i \cup S)$ there is a path in $G_k \setminus S$ between u and some vertex of V_i. Assume first that $u \in V_t$ for some $t \in [k-1] \setminus \{i,j\}$. As in Case 1, $(G_k \setminus S)[V_t]$ is connected and $E_{G_k \setminus S}(V_t, V_i) \neq \emptyset$. It follows that the required path exists. Assume then that $u \in V_j$. If $d_{G_k}(u, V_i) > 0$, then there is nothing to prove since $S \cap V_i = \emptyset$. Assume then that $d_{G_k}(u, V_i) = 0$; it follows by Property (v) above that $d_{G_k}(u, V_t) > 0$ holds for every $t \in [k-1] \setminus \{i,j\}$. If $d_{G_k \setminus S}(u, V_t) > 0$ holds for some $t \in [k-1] \setminus \{i,j\}$, then the required path exists as $(G_k \setminus S)[V_t]$ is connected and, as previously shown, there is an edge of $G_k \setminus S$ between V_t and V_i. Assume then that $d_{G_k \setminus S}(u, V_t) = 0$ holds for every $t \in [k-1] \setminus \{i,j\}$. It follows by Property (ii) above that $d_{G_k}(u, V_j) \geq 3$ and thus $d_{G_k \setminus S}(u, V_j) \geq 1$. Let $w \in V_j \setminus S$ be a vertex such that $uw \in E_k$. If $d_{G_k}(w, V_i) > 0$, then the required path exists. Otherwise, since $|V_j| \geq 5$ by Property (i) above, it follows by Property (vi) above that there exists a vertex $z \in N_{G_k \setminus S}(u, V_j) \cup N_{G_k \setminus S}(w, V_j)$ such that $d_{G_k}(z, V_i) > 0$. Hence, the required path exists.

We conclude that G_k is k-vertex-connected. $\square$

Note that while $\mathcal{G}_k$ includes very dense graphs, such as K_n, for every $k \geq 3$ and every sufficiently large n, this family also includes graphs with $\lceil kn/2 \rceil$ edges; that is, k-vertex-connected graphs which are as sparse as possible. One illustrative example of such a graph consists of $k-1$ pairwise vertex disjoint cycles, each of length $n/(k-1)$ where every pair of cycles is connected by a perfect matching (in particular, $k-1 \mid n$). The graphs Maker and Red will build in the proofs of Theorems 1.1 and 1.3 respectively, are fairly similar to this example.

3 Auxiliary games

In this section we consider several simple games. Some might be interesting in their own right whereas others are artificial. The results we prove about these games will be used in our proofs of Theorems 1.1 and 1.3. We divide this section into several subsections, each discussing one game.

3.1 A large matching game

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph, let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$, and let d be a positive integer. The board of the weak game $G(V_1, U_1; V_2, U_2; d)$ is E. Maker wins this game if and only if he accomplishes all of the following goals:

(i) Maker's graph is a matching.

(ii) $d_M(u) = 1$ for every $u \in (V_1 \setminus U_1) \cup (V_2 \setminus U_2)$.

(iii) $d_M(u) = 1$ for every $u \in V_1 \cup V_2$ for which $d_B(u) \geq d$.

(iv) $|\{u \in U_1 : d_M(u) = 0\}| \geq |U_1|/2$ and $|\{u \in U_2 : d_M(u) = 0\}| \geq |U_2|/2$.

Lemma 3.1 *Let m be a non-negative integer, let d be a positive integer, let $8d^{-1} \leq \varepsilon \leq 0.1$ be a real number and let $n_0 = n_0(m, d, \varepsilon)$ be a sufficiently large integer. Let $G = (V_1 \cup V_2, E)$ be a bipartite graph which satisfies all of the following properties:*

(P1) $n_0 \leq |V_1| \leq |V_2| \leq (1+\varepsilon)|V_1|$.

(P2) $d_G(u, V_2) \geq |V_2| - m$ *for every* $u \in V_1$.

(P3) $d_G(u, V_1) \geq |V_1| - m$ *for every* $u \in V_2$.

Let $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ be such that $\varepsilon|V_1| \leq |U_1| \leq 2\varepsilon|V_1|$ and $\varepsilon|V_2| \leq |U_2| \leq 2\varepsilon|V_2|$. Then Maker (as the first or second player) has a winning strategy for the game $G(V_1, U_1; V_2, U_2; d)$.

Proof First we describe a strategy for Maker and then prove it is a winning strategy. At any point during the game, if Maker is unable to follow the proposed strategy, then he forfeits the game.

Throughout the game, Maker maintains a matching M_G and a set $D \subseteq V_1 \cup V_2$ of *dangerous* vertices. A vertex $v \in V_1 \cup V_2$ is called dangerous if $d_M(v) = 0$ and $d_B(v) \geq d$. Initially, $M_G = D = \emptyset$.

For every positive integer j, Maker plays his jth move as follows.

(1) If $D \neq \emptyset$, then Maker claims an arbitrary free edge $uv \in E$ for which $u \in D$ and $d_M(v) = 0$. Subsequently, he updates $M_G := M_G \cup \{uv\}$ and $D := D \setminus \{u, v\}$.

(2) Otherwise, if there exists a free edge $uv \in E$ such that $u \in V_1 \setminus U_1$, $v \in V_2 \setminus U_2$ and $d_M(u) = d_M(v) = 0$, then Maker claims it. Subsequently, he updates $M_G := M_G \cup \{uv\}$.

(3) Otherwise, if there exists a vertex $u \in (V_1 \setminus U_1) \cup (V_2 \setminus U_2)$ such that $d_M(u) = 0$, then Maker claims a free edge $uv \in E$ such that $d_M(v) = 0$. Subsequently, he updates $M_G := M_G \cup \{uv\}$.

The game is over as soon as M_G covers $(V_1 \setminus U_1) \cup (V_2 \setminus U_2)$ and $D = \emptyset$.

It remains to prove that Maker can indeed follow the proposed strategy and that, by doing so, he wins the game $G(V_1, U_1; V_2, U_2; d)$.

It readily follows from its description that Maker can follow part (2) of the proposed strategy. Moreover, it is evident that Maker's graph is a matching at any point during the game. Hence, (even if he is forced to forfeit the game) he accomplishes goal (i). It follows that this game lasts at most $|V_1|$ moves. In particular, Breaker can create at most $2|V_1|/d \leq \varepsilon|V_1|/4 \leq \min\{|U_1|/4, |U_2|/4\}$ dangerous vertices throughout the game. Since Maker decreases the size of D whenever he follows part (1) of his strategy, we conclude that he follows this part at most $\min\{|U_1|/4, |U_2|/4\}$ times. Whenever Maker follows part (3) of the proposed strategy, $D = \emptyset$ and there is no free edge $uv \in E$ such that $u \in V_1 \setminus U_1$, $v \in V_2 \setminus U_2$ and $d_M(u) = d_M(v) = 0$. It follows by these two conditions and by Properties (P2) and (P3) that M_G covers at least $|V_1 \setminus U_1| - m - d$ of the vertices of $V_1 \setminus U_1$ and at least $|V_2 \setminus U_2| - m - d$ of the vertices of $V_2 \setminus U_2$. Since Maker matches a vertex of $(V_1 \setminus U_1) \cup (V_2 \setminus U_2)$ whenever he follows part (3) of the proposed strategy, we conclude that he follows this part at most $2(m+d) \leq \min\{|U_1|/4, |U_2|/4\}$

times. Since, moreover, Maker does not match any vertex of $U_1 \cup U_2$ when following part (2), we conclude that he matches at most $\min\{|U_1|/2, |U_2|/2\}$ vertices of $U_1 \cup U_2$ throughout the game. It follows that Maker accomplishes goal (iv). In particular, Maker can follow part (3) of the proposed strategy. Finally, since Maker accomplishes goal (iv), since Breaker creates at most $\min\{|U_1|/4, |U_2|/4\}$ dangerous vertices throughout the game, since Maker plays according to part (1) of the proposed strategy whenever $D \neq \emptyset$ and since he decreases $|D|$ whenever he does so, we conclude that Maker can follows part (1) of the proposed strategy. It now follows that Maker accomplishes goals (ii) and (iii) as well and thus wins the game. $\square$

3.2 A weak positive minimum degree game

In this subsection we study the weak *positive minimum degree* game $(E(G), \mathcal{D}_G^1)$, played on the edge set of some given graph G. The family of winning sets $\mathcal{D}_G^1$, consists of the edge sets of all spanning subgraphs of G with minimum degree at least 1. The following result was proved in [5].

Theorem 3.2 ([5] Corollary 1.3) *For sufficiently large n, Maker has a strategy for winning the weak game $(E(K_n), \mathcal{D}_{K_n}^1)$ within $\lfloor n/2 \rfloor + 1$ moves.*

We strengthen Theorem 3.2 by proving that its assertion holds even when the board is not complete, though still very dense.

Theorem 3.3 *For every positive integer m there exists an integer $n_0 = n_0(m)$ such that, for every $n \geq n_0$ and for every graph $G = (V, E)$ on n vertices with minimum degree at least $n - m$, Maker (as the first or second player) has a strategy for winning the weak positive minimum degree game $(E(G), \mathcal{D}_G^1)$, within at most $\lfloor n/2 \rfloor + 1$ moves.*

Proof We prove Theorem 3.3 by induction on m. At any point during the game, let $V_0 := \{u \in V : d_M(u) = 0\}$ denote the set of vertices of G which are isolated in Maker's graph and let $H := (B \cup (K_n \setminus G))[V_0]$.

In the induction step we will need to assume that $m \geq 3$. Hence, we first consider the cases $m = 1$ and $m = 2$ separately. If $m = 1$, then $G = K_n$ and thus the result follows immediately by Theorem 3.2. Assume then that $m = 2$ and assume for convenience that n is even (the proof for odd n is similar and in fact slightly simpler; we omit the straightforward details). For every $1 \leq i \leq n/2 - 1$, in his ith move, Maker claims a free edge uv such that $u, v \in V_0$ and $d_H(u) = \Delta(H)$. In each of his next two moves, Maker claims a free edge xy such that $x \in V_0$ and $y \in V$.

It is evident that, by following this strategy, Maker wins the positive minimum degree game $(E(G), \mathcal{D}_G^1)$, within $\lfloor n/2 \rfloor + 1$ moves. It thus remains to prove that he can indeed follow it. We prove that he can and that $\Delta(H) \leq 1$ holds immediately before Breaker's ith move for every $1 \leq i \leq n/2 - 1$, by induction on i. This holds for $i = 1$ by assumption. Assume it holds for some i. Clearly $\Delta(H) \leq 2$ holds immediately after Breaker's $(i+1)$st move. Moreover, there are at most two vertices $w \in V_0$ such that $d_H(w) = 2$ and if there are exactly two such vertices, then they are connected by an edge of Breaker. In his $(i+1)$st move, Maker claims an

edge which is incident with a vertex of maximum degree in H. It follows that $\Delta(H) \leq 1$ holds immediately after this move. Moreover, since $|V_0| = n - 2i \geq 4$ and $\Delta(H) \leq 2$ hold prior to this move, Maker can indeed play his $(i+1)$st move according to the proposed strategy. It is clear that Maker can play his $n/2$th and $(n/2+1)$st moves according to the proposed strategy.

Assume then that $m \geq 3$ and that the assertion of the theorem holds for $m-1$. We present a fast winning strategy for Maker. If at any point during the game Maker is unable to follow the proposed strategy, then he forfeits the game. The strategy is divided into the following two stages.

Stage I: Maker builds a matching while trying to decrease $\Delta(H)$. In every move, Maker claims a free edge uv such that $u, v \in V_0$, $d_H(u) = \Delta(H)$ and $d_H(v) = \max\{d_H(w) : w \in V_0, uw \in E(G \setminus B)\}$. The first stage is over as soon as $\Delta(H) \leq m-2$ first holds.

Stage II: Maker builds a spanning subgraph of $G[V_0]$ with positive minimum degree within $\lfloor |V_0|/2 \rfloor + 1$ moves.

It is evident that, if Maker can follow the proposed strategy without forfeiting the game, then he wins the positive minimum degree game on G within $\lfloor n/2 \rfloor + 1$ moves. It thus suffices to prove that he can indeed do so. First we prove that Maker can follow Stage I of his strategy, and moreover, that this stage lasts at most $\frac{(m-1)n}{2m} + 2$ moves. It is clear from the description of Maker's strategy that the following property is maintained throughout Stage I.

(∗) $\Delta(H) \leq m$ holds after every move of Breaker. Moreover, there are at most two vertices $u \in V_0$ such that $d_H(u) = m$ and if there are exactly two such vertices, then they are connected by an edge of Breaker.

For every non-negative integer i, immediately after Breaker's $(i+1)$st move, let $D(i) := \sum_{v \in V_0} d_H(v)$. Note that $D(i) \geq 0$ for every i and that $D(0) \leq (m-1)n + 2$ (before the game starts the maximum degree of H is at most $m-1$ and Breaker claims one edge in his first move). For an arbitrary non-negative integer i, let uv be the edge claimed by Maker in his $(i+1)$st move. At the time it was claimed, we had $d_H(u) = \Delta(H) \geq m-1$. Assume first that $d_H(v) \geq 2$ was true as well. It follows that $D(i+1) \leq D(i) - (m-1) - (m-1) - 2 - 2 + 2 = D(i) - 2m$ (we subtract $2m+2$ from $D(i)$ because of u, v and their neighbors, and then add 2 because Breaker claims some edge in his $(i+2)$nd move). It follows that there can be at most $\frac{(m-1)n}{2m}$ such moves throughout the first stage. Assume next that $d_H(v) \leq 1$; note that this entails $d_H(v) \leq m-2$ as $m \geq 3$ by assumption. It follows by Maker's strategy that u is connected by an edge of H to every vertex $x \in V_0$ such that $d_H(x) \geq 2$. Claiming uv decreases $d_H(w)$ by at least 1 for every $w \in V_0 \cap N_H(u)$. It follows by Property (∗) that after this move of Maker there is at most one vertex $z \in V_0$ such that $d_H(z) \geq m-1$. It is easy to see that, unless he forfeits the game, Maker can ensure $\Delta(H) \leq m-2$ in his next move. It follows that Stage I lasts at most $\frac{(m-1)n}{2m} + 2$ moves as claimed. In particular, we have $|V_0| \geq n/m - 4 > m+1 \geq \Delta(H)+1$ and thus Maker can indeed follow Stage I of the proposed strategy without forfeiting the game.

Next, we prove that Maker can follow Stage II of the proposed strategy. Since the first stage lasts at most $\frac{(m-1)n}{2m} + 2$ moves, $|V_0| \geq n/m - 4 \geq n_0(m-1)$ holds at the beginning of Stage II. Hence, it follows by the induction hypothesis that Maker can win the positive minimum degree game on $(G \setminus B)[V_0]$ within $\lfloor |V_0|/2 \rfloor + 1$ moves as claimed. □

Remark 3.4 *The requirement $n/m - 4 \geq n_0(m-1)$ appearing in the the proof of Theorem 3.3*

shows that the assertion of this theorem holds even for $m = c \log n / \log\log n$, where $c > 0$ is a sufficiently small constant.

3.3 A strong positive minimum degree game

In this subsection we study the strong version of the *positive minimum degree* game $(E(G), \mathcal{D}^1_G)$. We prove the following result.

Theorem 3.5 *For every positive integer m there exists an integer $n_0 = n_0(m)$ such that, for every $n \geq n_0$ and for every graph $G = (V, E)$ on n vertices with minimum degree at least $n-m$, Red has a strategy for winning the strong positive minimum degree game $(E(G), \mathcal{D}^1_G)$, within at most $\lfloor n/2 \rfloor + 1$ moves.*

Proof Let $\mathcal{S}_G$ be Maker's strategy for the weak positive minimum degree game $(E(G), \mathcal{D}^1_G)$ whose existence is guaranteed by Theorem 3.3. If n is odd, then Red simply follows $\mathcal{S}_G$. It follows by Theorem 3.3 that Red builds a spanning subgraph of G with positive minimum degree in $\lfloor n/2 \rfloor + 1$ moves. Since there is no such graph with strictly less edges, it follows that Red wins the game. Assume then that n is even.

We describe a strategy for Red for the strong positive minimum degree game $(E(G), \mathcal{D}^1_G)$ and then prove it is a winning strategy. If, at any point during the game, Red is unable to follow the proposed strategy, then he forfeits the game. At any point during the game, let $V_0 := \{v \in V : d_R(v) = 0\}$. The strategy is divided into the following five stages.

Stage I: In his first move of this stage, Red claims an arbitrary edge $e_1 = u_1 v_1$. Let $f = xy$ denote the edge Blue has claimed in his first move; assume without loss of generality that $x \notin e_1$. Let $A = \{z \in V_0 : xz \notin E\} \cup \{y\}$. For every $i \geq 2$, immediately before his ith move in this stage, Red checks whether $\Delta(B) \geq 2$, in which case he skips to Stage V. Otherwise, Red checks whether $A \cap V_0 = \emptyset$, in which case Stage I is over and Red proceeds to Stage II. Otherwise, let $w \in A \cap V_0$ be an arbitrary vertex. In his ith move in this stage, Red claims a free edge ww' for some $w' \in V_0$.

Stage II: Let $H = (G \setminus B)[V_0 \setminus \{x\}]$ and let $\mathcal{S}_H$ be the winning strategy for Maker in the weak positive minimum degree game, played on $E(H)$, which is described in the proof of Theorem 3.3. Let r denote the total number of moves Red has played in Stage I. For every $r < i \leq 3n/8$, immediately before his ith move in this stage, Red checks whether $\Delta(B) \geq 2$, in which case he skips to Stage V. Otherwise, Red plays his ith move according to the strategy $\mathcal{S}_H$. Once Stage II is over, Red proceeds to Stage III.

Stage III: Let $H = (G \setminus B)[V_0 \setminus \{x\}]$ and let $\mathcal{S}_H$ be the winning strategy for Maker in the weak positive minimum degree game, played on $E(H)$, which is described in the proof of Theorem 3.3. For every $3n/8 < i \leq n/2 - 1$, Red plays his ith move according to the strategy $\mathcal{S}_H$. Once Stage III is over, Red proceeds to Stage IV.

Stage IV: Let $z \in V_0 \setminus \{x\}$. If $xz \in E$ is free, then Red claims it. Otherwise, in his next two moves, Red claims free edges xx' and zz' for some $x', z' \in V$. In either case, the game is over.

Stage V: Let $H = (G \setminus B)[V_0]$ and let $\mathcal{S}_H$ be the winning strategy for Maker in the weak positive minimum degree game, played on $E(H)$, which is described in the proof of Theorem 3.3. In this stage, Red follows $\mathcal{S}_H$ until the end of the game.

We first prove that Red can indeed follow the proposed strategy without forfeiting the game. We consider each stage separately.

Stage I: Since $\delta(G) \geq n - m$, it follows that $|A| \leq m$. Since, moreover, n is sufficiently large with respect to m, we conclude that Red can follow Stage I of the proposed strategy.

Stage II: At the beginning of this stage we have $|V_0 \setminus \{x\}| = n - 2r - 1 \geq 0.99n$ and $\delta((G \setminus B)[V_0 \setminus \{x\}]) \geq |V_0| - 1 - m - r \geq |V_0| - 2m - 2$. Since n is assumed to be sufficiently large with respect to m, it follows by Theorem 3.3 that the required strategy $\mathcal{S}_H$ exists and that Red can indeed follow it throughout this stage.

Stage III: At the beginning of this stage we have $|V_0 \setminus \{x\}| \geq n/4 - 1$. Moreover, since Red did not skip to Stage V, it follows that $\delta((G \setminus B)[V_0 \setminus \{x\}]) \geq |V_0| - m - 2$. Since n is assumed to be sufficiently large with respect to m, it follows by Theorem 3.3 that the required strategy $\mathcal{S}_H$ exists and that Red can indeed follow it throughout this stage.

Stage IV: If the edge xz is still free, then Red can clearly claim it. Otherwise, Red can claim a free edge incident with x and a free edge incident with z since clearly $\Delta(B) < n/2$.

Stage V: At the beginning of this stage we have $|V_0| \geq n/4$. Moreover, since Red has just skipped to Stage V, it follows that $\delta((G \setminus B)[V_0]) \geq |V_0| - m - 2$. Since n is assumed to be sufficiently large with respect to m, it follows by Theorem 3.3 that the required strategy $\mathcal{S}_H$ exists and that Red can indeed follow it throughout this stage.

Next, we prove that if Red follows the proposed strategy, then he wins the game within at most $n/2 + 1$ moves. If Red reaches Stage V of the proposed strategy, then the game lasts at most $n/2 + 1$ moves. Since Red reaches Stage V only after Blue wastes a move, it follows by Theorem 3.3that Red wins the game in this case. Assume then that Red never reaches Stage V of the proposed strategy. It is clear that, at the end of Stage I, Red's graph is a matching. Moreover, it follows by the proof of Theorem 3.3 that Red's graph is a matching at the end of Stages II and III as well. Moreover, it is clear that $x \in V_0$ holds at this point. Hence, at the beginning of Stage IV, we have $V_0 = \{x, z\}$ for some $z \in V$. Moreover, by Stage I of the proposed strategy we have $xz \in E$. If xz is free, then Red claims it and thus builds a perfect matching in $n/2$ moves; hence, he wins the game in this case. Otherwise, the game lasts $n/2+1$ moves. However, in this case xz was claimed by Blue and thus $d_B(x) \geq 2$. We conclude that Red wins the game in this case as well. This concludes the proof of the lemma. □

4 The Maker-Breaker k-vertex-connectivity game

In this section we prove Theorem 1.1. In our proof we will use the following immediate corollary of Theorem 1.1 from [6].

Corollary 4.1 *Given a positive integer n, let $\mathcal{H}_n^+$ be the family of all edge sets of Hamilton cycles with a chord of K_n. If n is sufficiently large, then Maker (as the first or second player) has a strategy for winning $\mathcal{H}_n^+$ in exactly $n + 1$ moves.*

Proof of Theorem 1.1: Assume that $k \geq 4$ (at the end of the proof we will indicate which small changes have to be made to include the case $k = 3$). We present a strategy for Maker and then prove it is a winning strategy. If at any point during the game Maker is unable to

follow the proposed strategy, then he forfeits the game. Moreover, if after claiming kn edges, Maker has not yet built a k-vertex-connected graph, then he forfeits the game (we will in fact prove that Maker can build such a graph much faster; however, the technical upper bound of kn will suffice for the time being). The proposed strategy is divided into the following four stages.

Stage I: Let $V(K_n) = V_1 \cup V_2 \cup \ldots \cup V_{k-1}$ be an arbitrary equipartition of $V(K_n)$ into $k-1$ pairwise disjoint sets, that is, $||V_i| - |V_j|| \leq 1$ and $V_i \cap V_j = \emptyset$ for every $1 \leq i \neq j \leq k-1$. For every $1 \leq i \leq k-1$, let $\mathcal{S}_i$ be a winning strategy for Maker in the game $\mathcal{H}^+_{|V_i|}$ played on $E(K_n[V_i])$ whose existence is ensured by Corollary 4.1. In this stage, Maker's goal is to build a Hamilton cycle of $K_n[V_i]$ with a chord for every $1 \leq i \leq k-1$ while limiting the degree of certain vertices in Breaker's graph. If Maker is unable to accomplish both goals within $2n$ moves, then he forfeits the game. For every vertex $v \in V(K_n)$, let $1 \leq i_v \leq k-1$ be the (unique) index such that $v \in V_{i_v}$. Throughout this stage, Maker maintains a set $D \subseteq V(K_n) \times [k-1]$ of *dangerous* pairs. A pair $(v, i) \in V(K_n) \times [k-1]$ is called *dangerous* if $v \notin V_i$, $d_B(v, V_i) \geq 0.9|V_i|$, $d_M(v, V_i) = 0$ and $d_M(v) < k$. Initially, $D = \emptyset$. For every positive integer j, let $e_j = uv$ denote the edge which has been claimed by Breaker in his jth move. Maker plays his jth move as follows.

(i) If $e_j \in E(V_i)$ for some $1 \leq i \leq k-1$ and $M[V_i]$ is not yet a Hamilton cycle (of $K_n[V_i]$) with a chord, then Maker responds in this board according to the strategy $\mathcal{S}_i$.

(ii) Otherwise, if $D \neq \emptyset$, let $(z, i) \in D$ be a dangerous pair such that $d_B(z, V_i) = \max\{d_B(w, V_\ell) : (w, \ell) \in D\}$. Maker claims a free edge zw such that $w \in V_i$ and $d_M(w, V_{i_z}) = 0$. Subsequently, Maker updates $D := D \setminus \{(z, V_i), (w, V_{i_z})\}$.

(iii) Otherwise, if there exists $x \in \{u, v\}$ such that $M[V_{i_x}]$ is not yet a Hamilton cycle with a chord, then Maker plays as follows. Let $y \in \{u, v\}$ be such that $d_B(y, V(K_n) \setminus V_{i_y}) = \max\{d_B(v, V(K_n) \setminus V_{i_v}), d_B(u, V(K_n) \setminus V_{i_u})\}$ and let $z \in \{u, v\} \setminus \{y\}$. If $M[V_{i_y}]$ is not yet a Hamilton cycle with a chord, then Maker follows $\mathcal{S}_{i_y}$ on the board $E(V_{i_y})$, otherwise he follows $\mathcal{S}_{i_z}$ on $E(V_{i_z})$.

(iv) Otherwise, Maker plays according to $\mathcal{S}_i$ in a board $E(V_i)$ for some $1 \leq i \leq k-1$ such that $M[V_i]$ is not yet a Hamilton cycle with a chord.

As soon as $M[V_i]$ is a Hamilton cycle with a chord for every $1 \leq i \leq k-1$ and $D = \emptyset$, this stage is over and Maker proceeds to Stage II.

Stage II: Let C be the set of endpoints of the chords of $\bigcup_{i=1}^{k-1} M[V_i]$. At any point during this stage, let $Y_C := \{v \in C : d_M(v) < k\}$, let $Y_D := \{v \in V(K_n) : d_M(v) < k \text{ and } d_B(v) \geq k^{10}\}$ and let $Y := Y_C \cup Y_D$. For as long as $Y \neq \emptyset$, Maker picks an arbitrary vertex $v \in Y$ and plays as follows. Let $t = d_M(v)$ and let $\{i_1, \ldots, i_{k-t}\} \subseteq [k-1] \setminus \{i_v\}$ be $k-t$ distinct indices such that $d_M(v, V_{i_j}) = 0$ for every $1 \leq j \leq k-t$. In his next $k-t$ moves, Maker claims $k-t$ free edges $\{vv_{i_j} : 1 \leq j \leq k-t\}$ such that $v_{i_j} \in V_{i_j}$ and $d_M(v_{i_j}, V_{i_v}) = 0$ for every $1 \leq j \leq k-t$.

As soon as $Y = \emptyset$, this stage is over and Maker proceeds to Stage III.

Stage III: For every $1 \leq i \neq j \leq k-1$, let $A_{ij} \subseteq V_i$ denote the set of vertices $v \in V_i$ such that $d_M(v) < k$ and $d_M(v, V_j) = 0$. Moreover, for every $1 \leq i \neq j \leq k-1$, let $B_{ij} \subseteq A_{ij}$ be sets which satisfy all of the following properties:

(P1) $B_{ij} \cap B_{i\ell} = \emptyset$ for every $1 \le i \le k-1$ and for every $1 \le j \ne \ell \le k-1$.

(P2) $n/k^6 \le |B_{ij}| \le 2n/k^6$ for every $1 \le i \ne j \le k-1$.

(P3) $dist_{M[V_i]}(u,v) \ge 2$ for every $1 \le i \le k-1$ and for every two distinct vertices $u, v \in \bigcup_{j \in [k-1] \setminus \{i\}} B_{ij}$.

For every $1 \le i < j \le k-1$ let $G_{ij} = (A_{ij} \cup A_{ji}, E_{K_n \setminus B}(A_{ij}, A_{ji}))$ and let $\mathcal{S}_{ij}$ be the winning strategy for Maker in the game $G_{ij}(A_{ij}, B_{ij}; A_{ji}, B_{ji}; 2k^{10})$ which is described in the proof of Lemma 3.1.

At any point during this stage, for every $1 \le i < j \le k-1$, Maker maintains a matching M_{ij} of the board $E(G_{ij})$ and a set $D \subseteq V(K_n)$ of *dangerous* vertices. A vertex $v \in V(K_n)$ is called dangerous if $v \in B_{ij}$ for some $1 \le i \ne j \le k-1$ (without loss of generality assume $i < j$) and, moreover, v satisfies all of the following properties:

(1) v is not matched in M_{ij}.

(2) M_{ij} covers $(A_{ij} \setminus B_{ij}) \cup (A_{ji} \setminus B_{ji})$.

(3) $d_B(v) \ge k^{10}$.

Initially, $D = M_{ij} = \emptyset$ for every $1 \le i < j \le k-1$.

Let r denote the number of moves Maker has played throughout Stages I and II. For every $s > r$, let e_s denote the edge that has been claimed by Breaker in his sth move. Maker plays his sth move as follows:

(i) If $e_s \in E(G_{ij})$ for some $1 \le i < j \le k-1$ and M_{ij} does not yet cover $(A_{ij} \setminus B_{ij}) \cup (A_{ji} \setminus B_{ji})$, then Maker responds in the board $E(A_{ij}, A_{ji})$ according to the strategy $\mathcal{S}_{ij}$.

(ii) Otherwise, if $D \ne \emptyset$, then Maker claims a free edge uv between two sets B_{ij} and B_{ji} such that the following properties hold.

  (a) $u \in D$.

  (b) $d_B(u) = \max\{d_B(w) : w \in D\}$.

  (c) M_{ij} covers $(A_{ij} \setminus B_{ij}) \cup (A_{ji} \setminus B_{ji})$.

  Maker updates $D := D \setminus \{u, v\}$.

(iii) Otherwise, Maker picks arbitrarily $1 \le i < j \le k-1$ such that M_{ij} does not yet cover $(A_{ij} \setminus B_{ij}) \cup (A_{ji} \setminus B_{ji})$ and plays in the board $E(A_{ij}, A_{ji})$ according to the strategy $\mathcal{S}_{ij}$.

As soon as M_{ij} covers $(A_{ij} \setminus B_{ij}) \cup (A_{ji} \setminus B_{ji})$ for every $1 \le i < j \le k-1$ and $D = \emptyset$, this stage is over and Maker proceeds to Stage IV.

Stage IV: Let $U = \{v \in V(K_n) : d_M(v) = k-1\}$ and let $H := (K_n \setminus B)[U]$. Let $\mathcal{S}_H$ be a strategy for Maker for winning the positive minimum degree game $(E(H), \mathcal{D}_H^1)$ within $\lfloor |U|/2 \rfloor + 1$ moves. In this stage Maker follows $\mathcal{S}_H$ until $\delta(M) \ge k$ first occurs; at this point the game is over.

It is evident that if Maker can follow the proposed strategy without forfeiting the game, then, by the end of the game, he builds a graph $M \in \mathcal{G}_k$, which is k-vertex-connected by Proposition 2.1. It thus suffices to prove that Maker can indeed follow the proposed strategy without forfeiting the game and that, by doing so, he builds an element of $\mathcal{G}_k$ within $\lfloor kn/2 \rfloor + 1$ moves.

Our first goal is to prove that Maker can indeed follow the proposed strategy without forfeiting the game. We consider each stage separately.

Stage I: Since $|V_i| \geq \lfloor n/(k-1) \rfloor$ for every $1 \leq i \leq k-1$ and since n is sufficiently large with respect to k, it follows by Corollary 4.1 that Maker can follow part (i) of the proposed strategy for this stage.

Recall that, by definition, this stage lasts at most $2n$ moves and that $d_B(v, V_i) \geq 0.9|V_i| \geq 0.9n/k$ holds for every dangerous pair $(v,i) \in D$. Therefore, throughout Stage I, Breaker can create at most $4n/\left(\frac{0.9n}{k}\right) \leq 5k$ such pairs. We claim that at any point during Stage I, $d_B(v, V_i) \leq 0.95|V_i|$ holds for every vertex $v \in V(K_n)$ and every $i \in [k-1] \setminus \{i_v\}$. This is immediate by the definition of D for every pair $(v,i) \in (V(K_n) \times [k-1]) \setminus D$. Consider a point during this stage where $D \neq \emptyset$ (if this never happens, then there is nothing left to prove). If Breaker plays in $\bigcup_{i=1}^{k-1} E(V_i)$, then he does not increase $d_B(v, V_i)$ for any pair $(v,i) \in D$. Otherwise, Maker follows part (ii) of the proposed strategy for this stage and thus decreases the size of D. It follows that, throughout Stage I, Maker follows part (ii) of the proposed strategy at most $5k$ times. Since n is sufficiently large with respect to k, it follows that, throughout Stage I, $d_B(v, V_i) \leq 0.9|V_i| + 5k \leq 0.95|V_i|$ holds for every $v \in V(K_n)$ and every $i \in [k-1] \setminus \{i_v\}$ as claimed. Since Maker follows part (ii) of the proposed strategy at most $5k$ times and since he only claims edges of $\bigcup_{i=1}^{k-1} E(V_i)$ when following parts (i), (iii) or (iv) of the strategy, it follows that, throughout Stage I, $|\{u \in V_i : d_M(u, V_j) = 0\}| \geq 0.99|V_i|$ holds for every $1 \leq i \neq j \leq k-1$. Hence, Maker can follow part (ii) of the proposed strategy for this stage without forfeiting the game.

Finally, it readily follows from Corollary 4.1 that Maker can follow parts (iii) and (iv) of the proposed strategy for this stage.

It thus suffices to prove that Maker can achieve his goals for this stage within at most $2n$ moves. This readily follows from the following three simple observations.

(a) According to Corollary 4.1, for every $1 \leq i \leq k-1$, Maker can build a Hamilton cycle of $K_n[V_i]$ with a chord in $|V_i| + 1$ moves.

(b) Whenever Maker follows parts (i), (iii) or (iv) of the proposed strategy for this stage, he plays according to $\mathcal{S}_i$ for some $1 \leq i \leq k-1$.

(c) As previously noted, Maker follows part (ii) of the proposed strategy at most $5k$ times.

It follows that Stage I lasts at most $\sum_{i=1}^{k-1} (|V_i| + 1) + 5k = n + (k-1) + 5k < 2n$ moves.

We conclude that Maker can follow the proposed strategy for this stage, including the time limits it sets, without forfeiting the game.

Stage II: Since the entire game lasts at most kn moves, it follows that $|\{u \in V(K_n) : d_B(u) \geq k^{10}\}| \leq 2kn/k^{10}$ holds at any point during the game. Hence, $|Y| \leq 2(k-1) + 2n/k^9 \leq 3n/k^9$ holds at any point during this stage. Since $D = \emptyset$ at the end of Stage I and since Maker

spends at most k moves on every vertex of Y, it follows that, at any point during this stage, $d_B(v, V_i) \leq 0.9|V_i| + 3n/k^8 \leq 0.95|V_i|$ holds for every vertex $v \in Y$ and for every $i \in [k-1] \setminus \{i_v\}$. Since, as noted above, $|\{u \in V_i : d_M(u, V_j) = 0\}| \geq 0.99|V_i|$ holds for every $1 \leq i \neq j \leq k-1$ at the end of Stage I, it follows that $|\{u \in V_i : d_M(u, V_j) = 0\}| \geq 0.98|V_i|$ holds for every $1 \leq i \neq j \leq k-1$ throughout Stage II. We conclude that Maker can follow the proposed strategy for this stage without forfeiting the game.

Stage III: For every $1 \leq i \leq k-1$, let $A_i := \{u \in V_i : d_M(u) = 2\}$. Since Maker follows Stages I and II of the proposed strategy, we conclude that $|A_i| \geq \lfloor n/(k-1) \rfloor - (k+1)(5k + 2(k-1) + 2n/k^9) \geq 0.9n/k$ holds for every such i. Moreover, since Stage II lasts at most $k|Y| \leq n/k^7$ moves, it follows that $||A_{ij}| - |A_{ji}|| \leq n/k^7$ holds for every $1 \leq i < j \leq k-1$.

For every $1 \leq i \leq k-1$, let $B_i \subseteq A_i$ be a set which satisfies $|B_i| \geq \lfloor |A_i|/2 \rfloor \geq |A_i|/3$ and $dist_{M[V_i]}(u, v) \geq 2$ for every $u, v \in B_i$ (one example of such a set is obtained by enumerating the elements of A_i according to their order of appearance on the Hamilton cycle of $K_n[V_i]$ and taking either all even indexed vertices or all odd indexed vertices). Let $B_i = B_i^{(1)} \cup \ldots \cup B_i^{(i-1)} \cup B_i^{(i+1)} \cup \ldots \cup B_i^{(k-1)}$ be an equipartition of B_i. For every $1 \leq i < j \leq k-1$ let $B_{ij} \subseteq B_i^{(j)}$ and $B_{ji} \subseteq B_j^{(i)}$ be chosen such that Property (P2) in the description of the proposed strategy for this stage holds. Note that Properties (P1) and (P3) hold as well by the construction of the B_i's and the $B_i^{(j)}$'s.

Since, as noted above, $||A_{ij}| - |A_{ji}|| \leq n/k^7$ holds for every $1 \leq i < j \leq k-1$, since $d_B(u) < k^{10}$ holds for every $u \in A_i$ by Stage II of the proposed strategy and since n is sufficiently large with respect to k, it follows by Lemma 3.1 (with $\varepsilon = k^{-4}$) that Maker can follow parts (i) and (iii) of the proposed strategy for this stage.

Moreover, since $d_B(v) \geq k^{10}$ holds for every dangerous vertex and since the entire game lasts at most kn moves, it follows that Breaker can create at most $2kn/k^{10} \leq n/k^8$ such vertices. Since Maker spends exactly one move to treat a dangerous vertex and since $|B_{ij}| \geq n/k^6$ holds by construction for every $1 \leq i \neq j \leq k-1$, it follows that Maker can indeed follow part (ii) of the proposed strategy for this stage.

Stage IV: Whenever Maker follows part (ii) of the proposed strategy for this stage, he increases the degrees of two vertices by 1 each and decreases the size of D. Since the entire game lasts at most kn moves and since $d_B(v) \geq k^{10}$ holds for every $v \in D$, it follows that Maker follows part (ii) of the strategy at most $2n/k^9$ times. It follows by Lemma 3.1 and by Property (P2) that

$$|U| \geq \sum_{1 \leq i \neq j \leq k-1} |B_{ij}|/2 - 4n/k^9 \geq \binom{k-1}{2} n/(2k^6) - 4n/k^9 \geq n/(3k^6)\,.$$

Since n is sufficiently large with respect to k, it thus follows by Theorem 3.3 that Maker can follow the strategy $\mathcal{S}_H$ throughout this stage without forfeiting the game.

It remains to prove that, by following the proposed strategy, Maker wins the game within $\lfloor kn/2 \rfloor + 1$ moves. It follows by Theorem 3.3 that Stage IV lasts at most $\lfloor |U|/2 \rfloor + 1$ moves. It thus suffices to prove that $\delta(M) \leq k$ holds throughout Stages I, II and III. This follows quite easily from the description of Maker's strategy. There is one exception though. If $d_M(u, V_i) > 0$ for every $i \in [k-1] \setminus \{i_u\}$ and only then u becomes an endpoint of a chord, then we have $d_M(u) = k+1$. In order to overcome this problem, we include part (iii) of the strategy for Stage I. Recall that Maker follows part (ii) of the proposed strategy for this stage at most $5k$

times and that $||V_i| - |V_j|| \leq 1$ holds for every $1 \leq i < j \leq k-1$. It thus follows by part (iii) of the proposed strategy that if $d_B(u, V_i) \geq 0.9|V_i|$ and $d_B(u, V_j) \geq 0.9|V_j|$ hold for two distinct indices $i, j \in [k-1] \setminus \{i_u\}$, then $M[V_{i_u}]$ is already a Hamilton cycle with a chord; in particular we know whether u is an endpoint of this chord or not. Since $k \geq 4$, we can afford to wait until a vertex appears in two dangerous pairs. For $k = 3$ we have no choice but to ensure that if a vertex u satisfies $d_M(u, V_i)$ for $i \neq i_u$, then it will not become an endpoint of the chord of $M[V_{i_u}]$. In order to ensure this, one has to slightly alter Maker's strategy for the game $\mathcal{H}^+_{|V_{i_u}|}$. This can be done by adjusting the strategy given in the proof of Theorem 1.1 in [6] or the strategy given in the proof of Theorem 1.1 in [5] (the latter is easier). Note that this solution works for every $k \geq 3$. However, where possible, we preferred a solution which uses Maker's strategy for the Hamilton cycle with a chord game as a black box.

This concludes the proof of the theorem. □

5 The strong k-vertex-connectivity game

Proof of Theorem 1.3: Let $k \geq 3$ be an integer and assume first that kn is odd. Red simply follows Maker's strategy for the weak k-vertex-connectivity game $(E(K_n), \mathcal{C}_n^k)$ whose existence is guaranteed by Theorem 1.1. It follows by Theorem 1.1 that he builds a k-vertex-connected graph in $\lfloor kn/2 \rfloor + 1$ moves. Since, for odd kn, there is no graph G on n vertices such that $\delta(G) \geq k$ and $e(G) \leq \lfloor kn/2 \rfloor$, it follows that Red wins the strong k-vertex-connectivity game $(E(K_n), \mathcal{C}_n^k)$.

Assume then that kn is even. First, we present a strategy for Red and then prove it is a winning strategy. If at any point during the game Red is unable to follow the proposed strategy, then he forfeits the game. The proposed strategy is divided into the following two stages.

Stage I: Let $\mathcal{S}_M$ be the winning strategy for Maker in the weak game $(E(K_n), \mathcal{C}_n^k)$ which is described in the proof of Theorem 1.1. In this stage, Red follows Stages I, II and III of the strategy $\mathcal{S}_M$. As soon as Red first reaches Stage IV of $\mathcal{S}_M$, this stage is over and Red proceeds to Stage II.

Stage II: Let $U_0 := \{v \in V(K_n) : d_R(v) = k-1\}$ and let $G = (K_n \setminus B)[U_0]$. Let $\mathcal{S}_G$ be the winning strategy for Red in the strong positive minimum degree game $(E(G), \mathcal{D}_G^1)$ which is described in the proof of Theorem 3.5. We distinguish between the following three cases.

(1) If $\Delta(B) > k$, then Red continues playing according to the strategy $\mathcal{S}_M$ until the end of the game. That is, he follows Stage IV of $\mathcal{S}_M$ until his graph first becomes k-vertex-connected.

(2) Otherwise, if $d_B(v) \leq k-1$ for every $v \in U_0$, then Red plays the strong positive minimum degree game $(E(G), \mathcal{D}_G^1)$ according to the strategy $\mathcal{S}_G$ until his graph becomes k-vertex-connected.

(3) Otherwise, let $x \in U_0$ be a vertex such that $d_B(x) = k$. Let $H = G \setminus \{x\}$ and let $\mathcal{S}_H$ be the winning strategy for Red in the strong positive minimum degree game $(E(H), \mathcal{D}_H^1)$ which is described in the proof of Theorem 3.5. Let r denote the total number of moves Red has played so far. This case is further divided into the following four substages.

(i) For every $r < i \leq kn/2 - |U_0|/3$, immediately before his ith move in this stage, Red checks whether $\Delta(B) > k$, in which case he skips to Substage (iv). Otherwise, Red plays his ith move according to the strategy $\mathcal{S}_H$. As soon as this substage is over Red proceeds to Substage (ii).

(ii) For every $kn/2 - |U_0|/3 < i \leq kn/2 - 1$, Red plays his ith move according to the strategy $\mathcal{S}_H$. When this substage is over Red proceeds to Substage (iii).

(iii) Let $z \in U_0 \setminus \{x\}$ be a vertex of degree $k-1$ in Red's graph. If the edge $xz \in E(K_n)$ is free, then Red claims it. Otherwise, in his next two moves, Red claims free edges xx' and zz' for some $x', z' \in V(K_n)$. In both cases the game is over.

(iv) Let $U := \{v \in V(K_n) : d_R(v) = k-1\}$ and let $G' = (K_n \setminus B)[U]$. Let $\mathcal{S}_{G'}$ be the winning strategy for Red in the strong positive minimum degree game $(E(G'), \mathcal{D}^1_{G'})$ which is described in the proof of Theorem 3.5. In this substage, Red follows $\mathcal{S}_{G'}$ until the end, that is, until his graph first becomes k-vertex-connected.

It is evident that if Red can follow the proposed strategy without forfeiting the game, then, by the end of the game, he builds a graph $R \in \mathcal{G}_k$, which is k-vertex-connected by Proposition 2.1. It thus suffices to prove that Red can indeed follow the proposed strategy without forfeiting the game, that he builds an element of $\mathcal{G}_k$ within $\lfloor kn/2 \rfloor + 1$ moves and that he does so before $\delta(B) \geq k$ first occurs.

Our first goal is to prove that Red can indeed follow the proposed strategy without forfeiting the game. We consider each stage separately.

Stage I: Since n is sufficiently large with respect to k, it follows by Theorem 1.1 that Red can follow Stage I of the proposed strategy.

Stage II: We consider each of the three cases separately.

(1) Since Red has played all of his moves in Stage I according to the strategy $\mathcal{S}_M$, it follows by Theorem 1.1 that he can continue doing so until the end of the game.

(2) Since Red has played all of his moves in Stage I according to the strategy $\mathcal{S}_M$, it follows by the proof of Theorem 1.1 that $|U_0| = \Omega(n)$ holds at the beginning of Stage II. Since we are not in Case (1), it follows that $\delta(G) \geq |U_0| - k$. Since, moreover, n is sufficiently large with respect to k, it follows by Theorem 3.5 that Red can indeed follow the proposed strategy for this case without forfeiting the game.

(3) As previously noted, $|U_0| = \Omega(n)$ holds at the beginning of Stage II. Since we are not in Case (1), it follows that $\delta(H) \geq |U_0| - 1 - k$. Since, moreover, n is sufficiently large with respect to k, it follows by Theorem 3.5 that Red can follow Substage (i) of the proposed strategy for this case. Since $\Delta(B) \leq k$ holds at the beginning of Substage (ii) (otherwise Red would have skipped to Substage (iv)), it follows by an analogous argument that Red can follow Substage (ii) of the proposed strategy for this case as well. It follows by Substages (i) and (ii) of the proposed strategy that, at the beginning of Substage (iii), there are exactly two vertices of degree $k-1$ in Red's graph, one of which is x. Denote the other one by z. Since $\Delta(B) \leq k$ holds at the beginning of Substage (ii) and since this entire substage clearly lasts at most $|U_0|/3$ moves, it follows that $d_B(x) \leq k + |U_0|/3 < n/2$ and $d_B(z) \leq k + |U_0|/3 < n/2$ hold at the beginning of Substage (iii). Hence, Red can

follow Substage (iii) of the proposed strategy for this case. Finally, since $\Delta(B) \leq k+1$ and $|U| = \Omega(n)$ clearly hold at the beginning of Substage (iv) and since n is sufficiently large with respect to k, it follows by Theorem 3.5 that Red can follow Substage (iv) of the proposed strategy for this case.

It is evident from the description of the proposed strategy that the game lasts at most $kn/2+1$ moves. Hence, in order to complete the prove of the theorem, it suffices to show that, if the game lasts exactly $kn/2+1$ moves, then $\Delta(B) > k$. This clearly holds if the game ends in Case (1) or in Substage (iv) of Case (3). If the game ends in Case (2), then this follows by Theorem 3.5. Finally, if the game lasts exactly $kn/2+1$ moves and ends in Substage (iii) of Case (3), then $d_B(x) \geq k+1$ must hold.

This concludes the proof of the theorem. □

6 Concluding remarks and open problems

A more natural fastest possible strategy for the minimum-degree-k game. As noted in Corollary 1.2 (respectively Corollary 1.4), Maker (respectively Red) can win the weak (respectively strong) minimum-degree-k game $(E(K_n), \mathcal{D}_n^k)$ within $\lfloor kn/2 \rfloor + 1$ moves by following his strategy for the weak (respectively strong) k-vertex-connectivity game $(E(K_n), \mathcal{C}_n^k)$. While useful, this is not a very natural way to play this game. We have found a much more natural strategy for Maker (respectively Red) to win the weak (respectively strong) game $(E(K_n), \mathcal{D}_n^k)$ within $\lfloor kn/2 \rfloor + 1$ moves. It consists of two main stages. In the first stage, Maker (respectively Red) builds a graph with minimum degree $k-1$ and maximum degree k. This is done almost arbitrarily except that Maker (respectively Red) ensures that, if a vertex has degree $k-1$ in his graph, then its degree in Breaker's (respectively Blue's) graph will not be too large. In the second stage, he plays the weak (respectively strong) positive minimum degree game $(E(K_n), \mathcal{D}_n^1)$ on the graph induced by the vertices of degree $k-1$ in his graph. We omit the details.

Explicit winning strategies for other strong games. Following the observation made in [2] that fast winning strategies for Maker in a weak game have the potential of being upgraded to winning strategies for Red in the corresponding strong game, we have devised a winning strategy for Red in the strong k-vertex-connectivity game. It is plausible that one could devise a winning strategy for other strong games, where a fast strategy is known for the corresponding weak game. One natural candidate is the *specific spanning tree* game. This game is played on the edge set of K_n for some sufficiently large integer n. Given a tree T on n vertices, the family of winning sets $\mathcal{T}_n$ consists of all copies of T in K_n. It was proved in [3] that Maker has a strategy to win the weak game $(E(K_n), \mathcal{T}_n)$ within $n + o(n)$ moves provided that $\Delta(T)$ is not too large.

On the other hand, there are weak games for which Maker has a winning strategy and yet Breaker can refrain from losing quickly. Consider for example the *Clique* game $RG(n, q)$. The board of this game is the edge set of K_n and the family of winning sets consists of all copies of K_q in K_n. It is easy to see that for every positive integer q there exists an integer n_0 such that Maker (respectively Red) has a strategy to win the weak (respectively

strong) game $RG(n,q)$ for every $n \geq n_0$. However, it was proved in [1] that Breaker can refrain from losing this game within $2^{q/2}$ moves. The current best upper bound on the number of moves needed for Maker in order to win $RG(n,q)$ is $2^{2q/3} \cdot f(q)$, where $f(q)$ is some polynomial in q (see [4]). Note that this upper bound does not depend on the size of the board, in particular, it holds for an infinite board as well. Given that an exponential lower bound on the number of moves is known, it would be very interesting to find an explicit winning strategy for Red in the strong game $RG(n,q)$ for every positive integer q and sufficiently large n. Moreover, it would be interesting to determine whether Red can win this game on an infinite board.

References

[1] J. Beck, Ramsey Games, *Discrete Mathematics* 249 (2002), 3-30.

[2] A. Ferber and D. Hefetz, Winning strong games through fast strategies for weak games, *The Electronic Journal of Combinatorics* 18(1) (2011), P144.

[3] A. Ferber, D. Hefetz and M. Krivelevich, Fast embedding of spanning trees in biased Maker-Breaker games, *European Journal of Combinatorics*, 33 (2012), 1086-1099.

[4] H. Gebauer, On the clique-game, *European Journal of Combinatorics* 33(1) (2012), 8–19.

[5] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Fast winning strategies in Maker-Breaker games, *Journal of Combinatorial Theory, Ser. B.* 99 (2009), 39–47.

[6] D. Hefetz and S. Stich, On two problems regarding the Hamilton cycle game, *The Electronic Journal of Combinatorics* 16(1) (2009), R28.

[7] A. Lehman, A solution of the Shannon switching game, *J. Soc. Indust. Appl. Math.* 12 (1964), 687–725.

[8] D. B. West, **Introduction to Graph Theory**, Prentice Hall, 2001.